A New Approach to the Problem of Cascade Synthesis*

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SEVERAL familiar procedures in the synthesis of driving-point and transfer impedances yield a network in the form of a cascade of two-terminal-pair networks, each of which places in evidence a set of zeros of the even part of the driving-point impedance or of the transfer impedance. Thus each cycle in the Brune process produces a series resistance and a reactive two-terminal-pair network that provides a conjugate pair of even-part zeros on the j-axis; the total development consisting of a cascade of such networks separated by and terminated in appropriate resistances. The Darlington development for a driving-point impedance differs essentially only in that the removal of a series resistance is omitted at the beginning of each cycle, and correspondingly, the even-part zeros lie in the complex plane instead of being on the j-axis. The Dasher method of synthesizing the transfer impedance of an RC network follows an analogous development pattern and yields a cascade of component two terminal-pair networks of which each provides a complex pair of transmission zeros (in the left half-plane or on the j-axis).

The component network in each of these development procedures is usually thought of as representable (theoretically, at least) by a tee-circuit, and the typical cycle of computations consists essentially of determining in succession the three impedances (resp. admittances) of the tee-circuit and subtracting them from the given driving-point impedance or from the subsequently inverted remainder function. The appearance of negative elements in the tee-circuit thus found necessitates its conversion into an equivalent form in which such negative elements do not appear.

Not only is this computational process rather laborious, but because of the subtractions involved, one may lose numerical accuracy to such an extent as to require an excessively large number of significant figures at the outset in order to be able to carry through even a moderate number of cycles successfully.

In the following discussion, a more direct approach is suggested for carrying out a development of this kind that is not only theoretically more logical and straightforward, but obviates much of the computational tedium accompanying the conventional methods for accomplishing the same end result. The synthesis of an RC transfer impedance is initially chosen to serve as a means for illustrating character of alternate approach. Detailed modifications to accommodate Brune and Darlington development problems are indicated later.

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The Dasher Method of RC Synthesis

A typical cycle in this development is illustrated in Fig. 1. An input impedance and an over-all transfer impedance are given. The input impedance is decomposed into a two-terminal-pair network and a remainder function, such that the two-terminal-pair network produces a chosen pair of complex zeros of the given transfer impedance. In the figure, Z_1 is the input impedance, z_{11}, z_{22}, z_{12} are the open-circuit driving-point and transfer impedances of the component two terminal-pair network (also referred to as a "zero-section," since it places

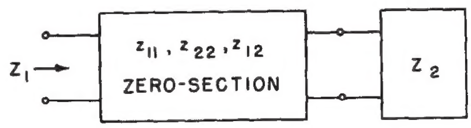

Fig. 1—A typical development cycle.

in evidence a pair of complex zeros of the given over-all transfer impedance), and Z_2 is the remainder function, having the same character as Z_1. The complete realization of the given impedances is obtained through carrying out a second cycle in terms of Z_2, and so forth, until only an open circuit or a constant is left as a remainder function.

The desired complex pair of transmission zeros is given by the quadratic factor

$$t(s) = (s - s_0)(s - \bar{s}_0) = (s + \sigma_0)^2 + \omega_0^2 \qquad (1)$$

with $s_0 = -\sigma_0 + j\omega$, $\bar{s}_0 = -\sigma_0 - j\omega_0$. The simplest form for the transfer impedance of the zero-section reads

$$z_{12} = \frac{Kt(s)}{s(s - s_a)} = K + \frac{k_0}{s} - \frac{k_{12}}{s - s_a} \qquad (K = \text{a real constant}) \quad (2)$$

since it must have two poles, and letting one of these coincide with the point $s=0$ leaves only one unknown pole $s = s_a$ to be determined. For reasons that depend upon the desired physical structure of the zero-section, we assume that its driving-point impedances have the form

$$z_{11} = K + \frac{k_0}{s} + \frac{k_{11}}{s - s_a} \qquad (3)$$

$$z_{22} = K + \frac{k_0}{s} + \frac{k_{22}}{s - s_a}. \qquad (4)$$

The essential difference between our present approach and the classical one already begins to become evident. Thus, instead of meekly inquiring what we must do to be able to produce a transmission zero, we use the direct approach of stating that we want the

zeros defined by (1) and the impedances (2), (3), and (4) to characterize the development cycle. Having stated our wants, we next see whether they are available at any price and if so what that price is. This query can only be answered in terms of a relation that connects all of the quantities shown in Fig. 1. For our purpose this relation is most appropriately written in the form

$$(z_{11} - Z_1)(z_{22} + Z_2) = z_{12}^2 = \frac{K^2 t^2(s)}{s^2(s - s_a)^2}. \tag{5}$$

Since the input impedance Z_1 is given, and the z_{sk}'s are already practically specified, the only unknown in this expression is Z_2. Our strategy in its evaluation is based upon the fact that we know the right-hand side of (5) (except for the values of K and s_a); we also know the factor $(z_{11}-Z_1)$ on the left. The other left-hand factor is, therefore, directly constructible, since in the product of the two left-hand factors everything must cancel except the constituent parts of z_{12}^2.

The driving-point impedance is now assumed in

$$Z_1(s) = A\frac{p(s)}{q(s)} + \frac{\rho_0}{s} + \rho_\infty \tag{6}$$

in which A, ρ_0, and ρ_∞ are constants, and

$$p(s) = (s - s_{p1})(s - s_{p2}) \cdots (s - s_{p n-1})$$
$$q(s) = (s - s_1)(s - s_2) \cdots (s - s_n)$$
$$s_n < s_{p(n-1)} < s_{n-1} < \cdots < s_{p1} < s_1 < 0. \tag{7}$$

We then recognize that the left-hand factors in (5) must have the specific forms

$$(z_{11} - Z_1) = \frac{K^2 t^2 h}{s(s - s_a)q} \tag{8}$$

$$(z_{22} + Z_2) = \frac{q}{s(s - s_a)h}. \tag{9}$$

Thus it is easy to see that the factor t^2 cannot be allotted to $(z_{22}+Z_2)$ because this is an RC driving-point impedance, and as such it cannot have complex zeros. The denominator in (8) is easily justified by inspection of the forms of z_{11} and Z_1, and h in the numerator of this factor is a polynomial of degree $n-2$, since the numerator must have the same degree as the denominator. The factor (9) is so chosen that (5) remains fulfilled.

Our whole problem of carrying through a typical development cycle is now reduced to the determination of one $n-2$ degree polynomial, which we shall write

$$h(s) = H(s - s_{h1})(s - s_{h2}) \cdots (s - s_{hn-2}). \tag{10}$$

The procedure for constructing this polynomial is as follows. From (6) and (8) we form

$$z_{11} = \frac{K^2 t^2 h + As(s - s_a)p}{s(s - s_a)q} + \frac{\rho_0}{s} + \rho_\infty. \tag{11}$$

Since z_{11} contains none of the nonzero poles of Z_1, we know that the numerator in the first term for z_{11} must be zero at the zeros of $q(s)$ which are $s = s_1, s_2, \cdots, s_n$. Hence we get

$$h = \frac{-As(s - s_a)p}{K^2 t^2} \qquad \text{for } s = s_1, s_2, \cdots, s_n. \tag{12}$$

In terms of the residues of Z_1 in its nonzero poles,

$$k_\nu = A\left[\frac{(s - s_\nu)p(s)}{q(s)}\right]_{s=s_\nu}, \tag{13}$$

(12) can be written

$$h(s_\nu) = \frac{-s_\nu(s_\nu - s_a)k_\nu}{K^2 t^2(s_\nu)}\left[\frac{q(s)}{s - s_\nu}\right]_{s=s_\nu}$$
$$\text{for } \nu = 1, 2, \cdots, n \tag{14}$$

and the Lagrangian interpolation formula then yields

$$h(s) = \frac{q(s)}{K^2}\sum_{\nu=1}^{n} m_\nu\left(\frac{s_\nu - s_a}{s - s_\nu}\right) \tag{15}$$

with the abbreviation

$$m_\nu = \frac{-s_\nu k_\nu}{t^2(s_\nu)}. \tag{16}$$

We now observe that (15) leads to a polynomial of degree $n-1$, whereas $h(s)$ should have the degree $n-2$. Hence the coefficient of the highest-power term in (15) must be zero. This gives

$$\sum_{\nu=1}^{n} m_\nu(s_\nu - s_a) = 0 \tag{17}$$

or

$$s_a = \frac{\sum m_\nu s_\nu}{\sum m_\nu} \tag{18}$$

from which the unknown nonzero pole in the impedances z_{sk} is determined. The coefficient of the next-highest power term in (15) is the constant multiplier H in the polynomial (10). In view of (18), this coefficient becomes

$$H = \frac{1}{K^2}\sum_{\nu=1}^{n} m_\nu s_\nu(s_\nu - s_a). \tag{19}$$

If we regard the positive quantities m_ν in (16) as a set of masses located at the points s_ν, then s_a in (18) may be interpreted as their center of gravity. It is thus easily clear that s_a has a negative real value as required. In order to see that the multiplier H as given by (19) is positive, we regard the quantities $m_\nu(s_\nu - s_a)$ as a set of forces which are negative to the left of s_a and positive to the right of this point, their sum being zero. Multiplication of these forces by $s_\nu < 0$, followed by addition, must lead to a positive value for H.[1]

[1] H may also be said to be a "second moment," since

$$\sum m_\nu s_\nu(s_\nu - s_a)^2 = \sum m_\nu(s_\nu - s_a)^2 + s_a \sum m_\nu(s_\nu - s_a)$$

in which the second term is zero by (17).

We now formulate the conditions necessary to obtain driving-point impedance having the form given by (3) and (4). In this connection we observe that it is sufficient if

$$\text{at } s=0 \qquad [sz_{11}]_{s=0} \geq k_0 \text{ and } [s(z_{22}+Z_2)]_{s=0} \geq k_0 \quad (20)$$

$$\text{and at } s=\infty \qquad [s_{11}]_{s=\infty} \geq K \text{ and } [z_{22}+Z_2]_{s=\infty} > k \quad (21)$$

for any excess in the value of each expression over the required minimum, may in the case of z_{11} be removed as a series branch preceding the zero section while in s_{22} it can be regarded as an allotment to the remainder function Z_2. In fact, the remainder should have such an allotment in order that it may have the same character as Z_1 hence the equality sign is not included in the conditions pertaining to the function $(s_{22}+Z_2)$.

If we observe from (2) that

$$k_0 = Kt(0)/(-s_a), \quad (22)$$

use of (9) and (11) enable us to write the conditions (20) and (21) in the form

$$\frac{K^2t^2(0)h(0)}{-s_a g(0)} + \rho_0 \geq \frac{Kt(0)}{-s_a}; \quad \frac{g(0)}{-s_a h(0)} > \frac{Kt(0)}{-s_a} \quad (23)$$

and

$$K^2H + \rho_\infty \geq K; \quad \frac{1}{H} > K \quad (24)$$

Noting that $t(0)=|s_0|^2$, and making use of (15) and (19), these conditions may be written respectively as

$$0 < K - |s_0|^2 \sum_{\nu=1}^{n} \frac{m_\nu}{-s_\nu}(s_\nu - s_a) \leq \frac{-s_a}{|s_0|^2}\rho_0 \quad (25)$$

and

$$0 < K - \sum_{\nu=1}^{n} m_\nu s_\nu(s_\nu - s_a) \leq \rho_\infty. \quad (26)$$

They have the algebraic form

$$\begin{aligned} 0 < K - a \leq b \\ 0 < K - c \leq d \end{aligned} \quad (27)$$

in which a, b, c, d are all positive. In order that a single positive K-value may satisfy them, it is found necessary and sufficient that

$$-b < (a-c) < d. \quad (28)$$

With reference to (25) and (26), and noting that $s_\nu < 0$, we have

$$(a-c) = |s_0| \sum_{\nu=1}^{n} \left\{ \left|\frac{s_0}{s_\nu}\right| + \left|\frac{s_\nu}{s_0}\right| \right\} m_\nu(s_\nu - s_a). \quad (29)$$

If we define a weighting function

$$w_\nu = \left|\frac{s_0}{s_\nu}\right| + \left|\frac{s_\nu}{s_0}\right| \quad (30)$$

and the weighted masses

$$\tilde{m}_\nu = w_\nu m_\nu \quad (31)$$

with their center of gravity

$$\bar{s}_a = \frac{\sum \tilde{m}_\nu s_\nu}{\sum \tilde{m}_\nu}, \quad (32)$$

then (29) may be written

$$(a-c) = |s_0| (\bar{s}_a - s_a) \sum \tilde{m}_\nu, \quad (33)$$

and the condition (28) becomes

$$-\left|\frac{s_a}{s_0}\right| \rho_0 < |s_0|^2(\bar{s}_a - s_a) \sum \tilde{m}_\nu < |s_0| \rho_\infty. \quad (34)$$

Since the left-hand quantity in this condition is negative and the right-hand one positive, we see that the center of gravity $\bar{s}_a$ of weighted masses may lie on either side of center of gravity s_a of unweighted masses.

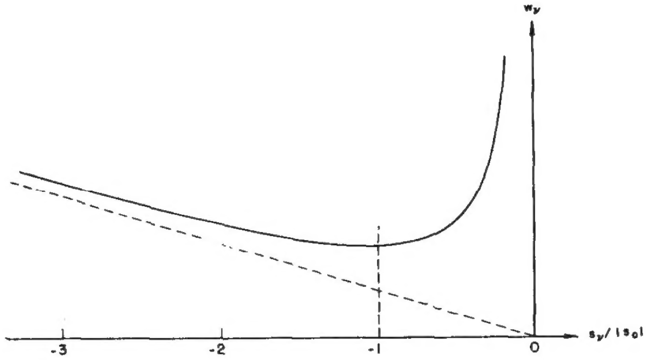

Fig. 2—The weighting function.

If we regard s_ν in the weighting function (30) as a continuous variable for the moment, then we see that this function has the character shown in Fig. 2. It is clear, therefore, that if $|s_0|$ is small compared with $|s_a|$, then $(\bar{s}_a - s_a)$ will be negative, while if $|s_0|$ is large compared with $|s_a|$, then $(\bar{s}_a - s_a)$ will be positive.[2] In absolute value, $(\bar{s}_a - s_a)$ cannot, however, exceed the value of $|s_n - s_a|$ for small $|s_0|$ nor of $|s_a - s_1|$ for large values of $|s_0|$; and for values of $|s_0|$ that are about the same as $|s_a|$, the value of $(\bar{s}_a - s_a)$ must be rather small since the weighting function will shift the center of gravity of the masses by only a small amount.

In considering the probability of being able to fulfill the condition (34), it is significant to observe the dependence of m_ν upon s_0 as expressed by (16) and (1). Thus for large values of $|s_0|$, $t^2(s_\nu) \approx |s_0|^4$ and hence m_ν varies inversely as $|s_0|^4$. Since w_ν, according to (30), varies as $|s_0|$ for large values of $|s_0|$, the weighted masses $\tilde{m}_\nu$ are seen to be inversely proportional to $|s_0|^3$. The central quantity in the condition (34), therefore, varies inversely as $|s_0|$ and hence becomes small but remains positive so that the right-hand inequality governs. Since the right-hand quantity is proportional to $|s_0|$, we see that the condition will easily be met.

For small values of $|s_0|$, on the other hand, (1) shows that $t^2(s_\nu) \approx |s_\mu|^4$. w_ν now varies inversely as $|s_0|$, and

[2] Recall that s_a and the s_ν are negative.

so does $\tilde{m}_\nu$ since m_ν is substantially independent of the value of $|s_0|$. The central quantity in the condition (34) becomes proportional to $|s_0|$ and again is small, but now it is negative so that the left-hand inequality governs. Since the left-hand quantity varies inversely as $|s_0|$, we see that the condition will easily be met for this case also.

Finally, if the value of $|s_0|$ is about the same as $|s_a|$ so that $(\bar{s}_a - s_a)$ is small compared with $|s_0|$, the condition is again likely to be fulfillable.

In this connection it is also significant to observe that the values of both ρ_0 and ρ_∞ can at any stage in the synthesis development be increased by the partial removal of a pole of the admittance $Y_1(s) = 1/Z_1(s)$. Thus, if for the moment we indicate a partial fraction expansion of $Z_1(s)$ by the expression

$$Z_1(s) = \frac{k_1}{s + \sigma_1} + \frac{k_3}{s + \sigma_3} + \cdots + \frac{\rho_0}{s} + \rho_\infty, \quad (35)$$

then the expansion of the corresponding admittance has the form

$$Y_1(s) = -\frac{k_2}{s + \sigma_2} - \frac{k_4}{s + \sigma_4} - \cdots + \frac{1}{\rho_\infty}, \quad (36)$$

in which the k_ν-values are positive. Since we know that $Y_1(0) = 0$, because $Z_1(s)$ has a pole at $s = 0$, we have

$$\frac{1}{\rho_\infty} = \frac{k_2}{\sigma_2} + \frac{k_4}{\sigma_4} + \cdots. \quad (37)$$

The partial fraction expansion of $Y_1(s)/s$ (in which the s-multiplied terms represent admittances of series RC branches) reads

$$\frac{Y_1(s)}{s} = \frac{k_2/\sigma_2}{s + \sigma_2} + \frac{k_4/\sigma_4}{s + \sigma_4} + \cdots. \quad (38)$$

It contains no constant term since it must be zero for $s = \infty$, and it contains no term representing a pole at $s = 0$ because $sZ_1(s)$, by (35), becomes equal to ρ_0 for $s = 0$. Hence (38) yields

$$\frac{1}{\rho_0} = \frac{k_2}{\sigma_2^2} + \frac{k_4}{\sigma_4^2} + \cdots. \quad (39)$$

From (37) and (39) we see that reduction of a k-value yields smaller values for both $1/\rho_0$ and $1/\rho_\infty$, and hence is a means for expanding the permissible range in the condition (34).

Returning to the conditions (25) and (26), we observe that fulfilling (25) with the equals sign (equivalent to making the residue of z_{11} at $s = 0$ equal to k_0) avoids the necessity of removing a series capacitance branch on the input side of the zero section. In terms of the form (27) of the conditions (25) and (26), this possibility is found to exist if

$$-b < (a - c) < d - b, \quad (40)$$

leading to the following extended form of (34)

$$-\left|\frac{s_a}{s_0}\right|\rho_0 < |s_0|^2(\bar{s}_a - s_a)\sum \tilde{m}_\nu < |s_0|\rho_\infty - \left|\frac{s_a}{s_0}\right|\rho_0. \quad (41)$$

For small values of $|s_0|$ for which $(\bar{s}_a - s_a)$ is negative, this condition is able to be fulfilled whenever (34) is; while for large values of $|s_0|$ for which the value of $|s_a/s_0|\rho_0$ is small, it is only moderately more restrictive than (34). For intermediate values of $|s_0|$, fulfillment of (41) may require the partial removal of a pole of $Y_1(s)$ as just described, but on the whole the possibility of avoiding the series capacitance without resorting to this expedient is rather good. In any event the possibility of trading the removal of a shunt branch for the removal of a series capacitance may be useful in certain situations.

Finally it is necessary to show that the remainder function $Z_2(s)$ is pr and RC. This will be so if $(z_{22} + Z_2)$ is pr and RC. From (9) we see that we need merely satisfy ourselves that the zeros of $q(s)$ and those of $s(s - s_a)h(s)$ separate each other. According to (12) we have

$$s(s - s_a)h(s) = \frac{-As^2(s - s_a)^2 p(s)}{K^2 t^2(s)} \text{ for } s = s_\nu. \quad (42)$$

This relation shows that the polynomial $s(s - s_a)h(s)$ has algebraic signs opposite to those of $p(s)$ at the zeros of $q(s)$. Since the zeros of $p(s)$ and $q(s)$ alternate, a simple sketch shows at once that those of $s(s - s_a)h(s)$ and of $q(s)$ must likewise alternate. Thus the pr and RC character of $(z_{22} + Z_2)$ is assured; and since the conditions (25) and (26) provide for the residue of $(z_{22} + Z_2)$ at $s = 0$ and its value at $s = \infty$ being respectively larger than those of z_{22}, it follows that the functions z_{22} and Z_2 are separately pr and RC.

At the pole at $s = s_a$, (2) shows that

$$k_{12} = Kt(s_a)/(-s_a) \quad (43)$$

is a positive quantity. The pr and RC character of z_{22} guarantees that its residue k_{22} (4) at $s = s_a$ is positive; and fulfillment of the residue condition

$$k_{11}k_{22} - k_{12}^2 = 0, \quad (44)$$

which is automatic since $Z_1(s)$ does not contain the pole at $s = s_a$, secures a positive value for the residue k_{11} of z_{11} at this point. The realizability conditions for the "zero section" and for the remainder function are thus established; and the condition (34), which as we have found to be fulfillable, assures the success of the development cycle in its contemplated form.

The computational procedure for determining the pertinent impedance functions is to expand $(z_{22} + Z_2)$ into partial fractions after the polynomial $h(s)$ is determined and its zeros as well as the value of s_a in (18) are found. The term for the pole at $s = s_a$ plus $(k_0/s) + K$ (with k_0 determined from (22) belong to z_{22}; the rest is $Z_2(s)$ with

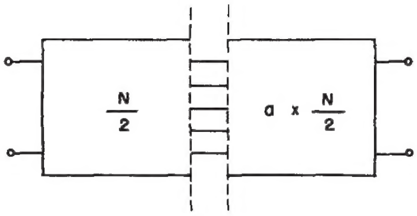

Fig. 3—Dissymmetrical network derivable from a symmetrical lattice.

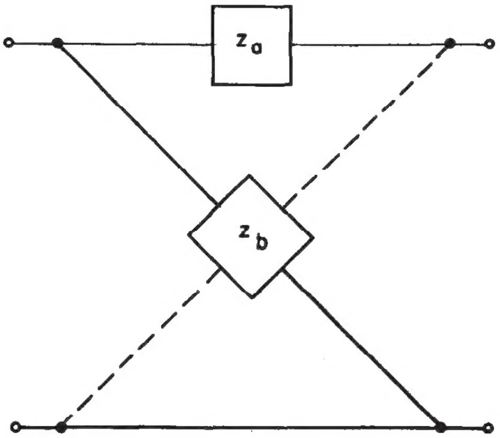

Fig. 4—Symmetrical lattice.

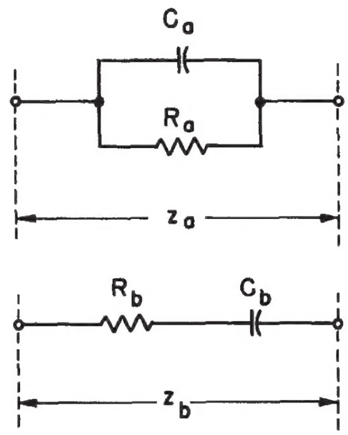

Fig. 5—Lattice impedances.

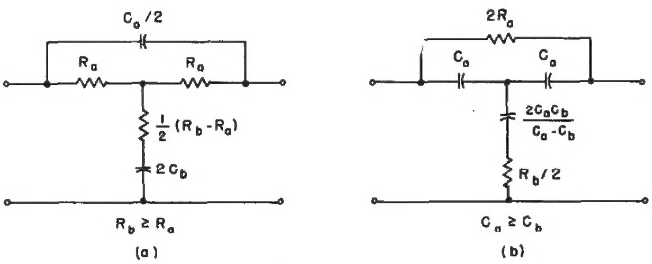

Fig. 6—Bridged-T equivalents of the lattice.

all of its residues computed, ready for carrying out the next cycle.[3] z_{12} is completely known from (2) and (43); and z_{11} is given by (3) is determined as soon as the residue k_{11} is computed from the condition (44).

We come now to the synthesis of the zero section from its z_{sk}'s. Although this component two terminal-pair network is not symmetrical, it is representable as shown in the sketch of Fig. 3, as the recombination of the two halves of a bisected symmetrical network after the right-hand half has its impedance level multiplied by the factor $a>0$. The symmetrical lattice (Fig. 4) from which this particular form of dissymmetrical network is derivable, has the impedances

$$z_a = \frac{z_{22} - z_{12}}{a}, \tag{45}$$

and

$$z_b = \frac{z_{22}}{a} + z_{12} \tag{46}$$

in which the impedance level factor is given by

$$a = \frac{z_{22} - z_{12}}{z_{11} - z_{12}}, \tag{47}$$

and the z_{sk}'s relate to the network of Fig. 3.

[3] For the constant K, any value is chosen that fulfills the conditions (25) and (26), preferably one that avoids the necessity of removing a series branch of the zero section.

Substitution from (2), (3) and (4) gives

$$a = \frac{k_{22} + k_{12}}{k_{11} + k_{12}} = \frac{k_{22}}{k_{12}} = \frac{k_{12}}{k_{11}}, \tag{48}$$

where use is made of the condition (44). The reason for choosing the z_{sk}'s in the form expressed by (2), (3) and (4) is now clear. Namely, these are the most general forms that the z_{sk}'s can have if the value of the impedance level factor is to be positive real. Since the k_{sk}'s in (48) have been shown to be positive real, such a value for the factor a is assured.

Substituting (2), (3), (4) into (45) and (46) yields

$$z_a = \frac{k_{11} + k_{12}}{(s - s_a)}, \tag{49}$$

$$z_b = \left(K + \frac{k_0}{s}\right) \times \frac{k_{11} + k_{12}}{k_{12}}. \tag{50}$$

The circuits for these impedances are shown in the sketches of Fig. 5, and the element values are

$$R_a = \frac{k_{11} + k_{12}}{-s_a}, \qquad C_a = \frac{1}{k_{11} + k_{12}}, \tag{51}$$

$$R_b = \frac{K(k_{11} + k_{12})}{k_{12}}, \qquad C_b = \frac{k_{12}}{k_0(k_{11} + k_{12})}. \tag{52}$$

The process of converting this lattice to an unbalanced form varies according to the relative element values (51) and (52). In Fig. 6 parts (a) and (b) are

shown respectively the bridged-T equivalents obtainable if $R_b/R_a \geqq 1$ or if $C_a/C_b \geqq 1$. When neither of these conditions is met, the procedure is to represent z_b as shown in Fig. 7 in which α is a positive numeric with a

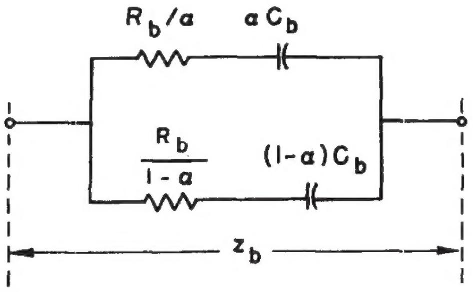

Fig. 7—Decomposition of the lattice cross-arm impedance.

value between zero and unity. As shown in Fig. 8, the lattice may now be regarded as a parallel combination of two simpler ones. Transformation of each of these to an equivalent tee-circuit yields the twin-tee shown in

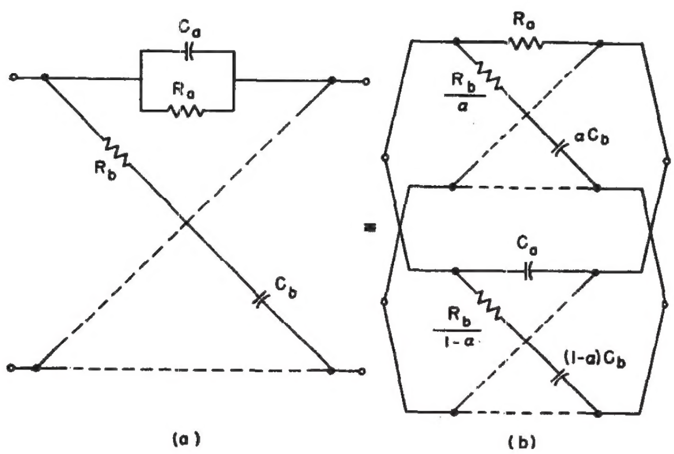

Fig. 8—Lattice and equivalent twin.

Fig. 9, and the condition upon the parameters (51) and (52) for which all resultant element values are positive is found to be

$$\frac{R_b}{R_a} + \frac{C_a}{C_b} \geqq 1. \tag{53}$$

It is clear that this condition is more lenient than either of the conditions for the existence of the bridged-T circuits of Fig. 6 in the sense that when neither one of these is fulfilled, the condition (52) may still be met.

Since the value of the numeric α can now be freely chosen within the range $0<\alpha<1$, it is possible to reduce by one the number of capacitances in the twin-tee Thus if we choose

$$\alpha = 1 - \frac{C_a}{C_b}, \tag{54}$$

the capacitance in the shunt branch of the lower tee in Fig. 9 is eliminated. The value (54) fails to be positive if $C_a/C_b > 1$, but then the bridged-T circuit of Fig. 6(b) is a possible realization.

Substitution from (51) and (52) converts the condition (53) into the form

$$\frac{k_0 - s_a K}{k_{12}} \geqq 1, \tag{55}$$

and through use of (22) and (43), noting that $t(0) = |s_0|^2$ we get

$$s_a^2 + |s_0|^2 \geqq t(s_a). \tag{56}$$

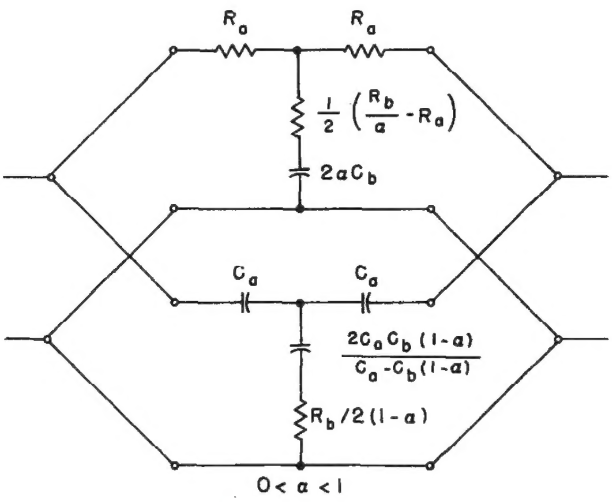

Fig. 9—Resulting twin-tee.

The interpretation of this result is shown in the s-plane sketch of Fig. 10, from which it is clear that the desired transmission zero is restricted to lie in the left half-plane or upon the j-axis.

In terms of the parameter values expressed by (51) and (52) it is collaterally of interest to see that the conditions for the relizability of the bridged-T networks of Fig. 6(a) and (b) are respectively expressible as

$$t(s_a) \leqq s_a^2, \qquad t(s_a) \leqq |s_0|^2. \tag{57}$$

The s-plane significance of each of these conditions is readily evident from the sketch of Fig. 10.

The desired zero section is finally formed through bisecting the pertinent unbalanced structure, multiplying the impedance level of the right-hand half by the factor a, (48), and rejoining the halves.

It remains to point out that a completely analogous procedure may be carried out on an admittance basis for which the development applies to an input admittance $Y_1(s)$ having the same form as $Z_1(s)$ in (6) except that the terms $(\rho_0/s)+\rho_\infty$ are replaced by $\rho_0+\rho_\infty s$. Here the component two terminal-pair network (zero section) is characterized in terms of the short-circuit driving-point and transfer admittances

$$-y_{12} = \frac{Kt(s)}{(s - s_a)} = k_0 + Ks - \frac{k_{12}s}{s - s_a}, \tag{58}$$

$$y_{11} = \cdots = k_0 + Ks + \frac{k_{11}s}{s - s_a}, \tag{59}$$

$$y_{22} = \cdots = k_0 + Ks + \frac{k_{22}s}{s - s_a}, \tag{60}$$

and the remainder function is an admittance $Y_2(s)$ having the same detailed character as $Y_1(s)$. Since every step in this development procedure on an admittance basis is the dual of the corresponding step in the process just described and all analytic expressions are essentially the same (except for some minor changes due to the behavior of the admittances at the points $s=0$ and $s=\infty$ being interchanged as compared with that of impedances), there is no need to carry through details.

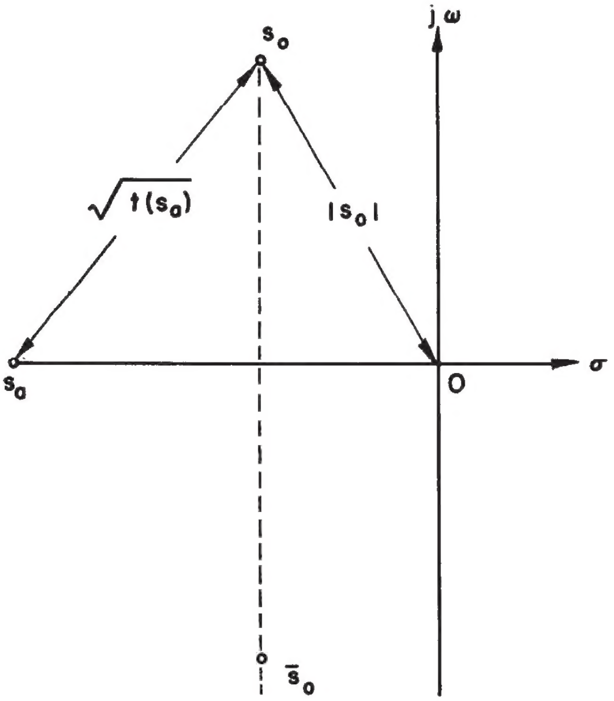

Fig. 10—Illustrating restriction of transmission zeros to left half-plane.

It is, however, pertinent to mention that use of the admittance basis may be made under either of two distinctly different situations, namely:

(a) We may be given the over-all short-circuit driving-point and transfer admittances $Y_{11}(s)$ and $Y_{12}(s)$ and be asked to find a corresponding two terminal-pair network, just as on the previously assumed impedance basis our problem was to associate a network with the over-all open-circuit driving-point and transfer impedances $Z_{11}(s)$ and $Z_{12}(s)$.

(b) We are given the impedances $Z_{11}(s)$ and $Z_{12}(s)$ and are asked to determine the pertinent two terminal-pair network, but $Z_{11}(s)$ has no pole at $s=0$ and its asymptotic value for $s\rightarrow\infty$ is zero, so that the terms with ρ_0 and ρ_∞ as in (6) are absent.

Although, in the latter event, we can always create a pole at $s=0$ through the removal of an appropriate shunt resistance, it may be simpler to reciprocate $Z_{11}(s)$ and proceed on an admittance basis *as though this reciprocated $Z_{11}(s)$ were the short-circuit driving-point admittance* which, of course, it isn't. Nevertheless, if we develop this Y_{11}-function into a cascade of component two terminal-pair networks characterized by the admittances (58), (59) and (60), choosing the same transmission zeros as those specified for the over-all transfer impedance $Z_{12}(s)$, then we will obtain a resultant development that is just as pertinent to the given $Z_{11}(s)$ and $Z_{12}(s)$ as we would if these had been developable on an impedance basis.

The truth of this statement follows from the fact that the development of $Y_{11}=1/Z_{11}$ on an admittance basis must just as surely yield the desired Z_{11}-function; and the z_{12}-function for any component network must have the same zeros as the y_{12}-function for that network because $z_{12}=-y_{12}/(y_{11}y_{22}-y_{12}^2)$ in which the y_{sk}'s fulfill the residue condition with the equals sign at all finite poles. Hence the development of $Y_{11}=1/Z_{11}$ on an admittance basis provides: (a) the correct input impedance, (b) the desired over-all transmission zeros, which is neither more nor less than is provided by a development of $Z_{11}(s)$ on an impedance basis.[4]

These ideas tell us that in the conditions (25) and (26), the left-hand inequalities can at any stage in the total development procedure be replaced by equality signs if we wish. By doing so, the remainder function $Z_2(s)$ will not have the terms with ρ_0 and ρ_∞ as does $Z_1(s)$ in (6). We cannot proceed further on an impedance basis in the normal manner, but we can reciprocate the remainder and proceed on an admittance basis. This ability to switch from one basis to another, perhaps repeatedly during an entire development, injects a degree of flexibility that may prove useful under certain circumstances.

For the synthesis of the zero-section on an admittance basis we have in place of (45) and (46) for the corresponding lattice admittances

$$y_a = y_{11} - ay_{12} \tag{61}$$

$$y_b = y_{11} + y_{12} \tag{62}$$

in which a is still considered to be an *impedance* level factor and is given by

[4] For proper behavior of the transfer impedance at 0 and ∞, the admittance development must end in a series resistance R, immediately before the short-circuit. (This is, of course, the usual situation.) Then, on the impedance basis, R becomes a terminal shunt branch, immediately before the open-circuit.

$$a = \frac{y_{11} + y_{12}}{y_{22} + y_{12}}. \tag{63}$$

Substitution from (58), (59) and (60) gives

$$a = \frac{k_{11} + k_{12}}{k_{22} + k_{12}} = \frac{k_{12}}{k_{22}} = \frac{k_{11}}{k_{12}}, \tag{64}$$

and for the lattice admittances

$$y_a = (k_0 + Ks)\left(1 + \frac{k_{11}}{k_{12}}\right), \tag{65}$$

$$y_b = \frac{(k_{11} \quad k_{12})s}{s - s_a}. \tag{66}$$

These again yield the networks of Fig. 5, but instead of the element values (51) and (52) we have

$$R_a = \frac{k_{12}}{k_0(k_{11} + k_{12})}, \qquad C_a = \frac{K(k_{11} + k_{12})}{k_{12}}, \tag{67}$$

$$R_b = \frac{1}{k_{11} + k_{12}}, \qquad C_b = \frac{k_{11} + k_{12}}{-s_a}. \tag{68}$$

The transformation methods for converting the lattice to an equivalent unbalanced structure and the conditions under which this is possible, are the same as before.

Remarks Regarding the Brune and Darlington Developments

We shall consider the Darlington cycle first since, like the RC problem just discussed, it also deals with complex transmission zeros. However, in contrast with the RC situation, we must consider not only the left half-plane zeros defined by $t(s)$ in (1), but also their images in the right half-plane. Altogether we have the quadruplet set of transmission zeros described by

$$t(s) = (s - s_0)(s - \bar{s}_0) = t_+$$
$$t(-s) = (s + s_0)(s + \bar{s}_0) = t_- \tag{69}$$

with

$$s_0 = -\sigma_0 + j\omega_0, \quad \bar{s}_0 = -\sigma_0 - j\omega_0, \quad \sigma_0 > 0, \quad \omega_0 > 0.$$

The given driving-point impedance we assume in the form

$$Z_1(s) = \frac{Ap(s)}{q(s)} = \frac{A(s - s_{p1}) \cdots (s - s_{pn})}{(s - s_1) \cdots (s - s_n)} = \frac{AP(s)}{Q(s)}, \tag{70}$$

in which

$$P(s) = p(s)t(s) \quad \text{and} \quad Q(s) = q(s)t(s) \tag{71}$$

are the numerator and denominator polynomials after $Z_1(s)$ has been appropriately augmented so as to insure that its even-part zeros are of double order, as is required in the normal Darlington procedure.

The z_{sk}'s of the lossless two terminal-pair network are assumed in the simplest form

$$Z_{12} = \frac{Kt(s)t(-s)}{s(s^2 - s_a^2)} = K_{12}s + \frac{k_0}{s} - \frac{2k_{12}s}{s^2 - s_a^2} \tag{72}$$

$$z_{11} = \cdots = K_{11}s + \frac{k_0}{s} + \frac{2k_{11}s}{s^2 - s_a^2} \tag{73}$$

$$z_{22} = \cdots = K_{22}s + \frac{k_0}{s} + \frac{2k_{22}s}{s^2 - s_a^2} \tag{74}$$

in which the K_{jk} and the k_{jk} are to obey the perfect coupling law. It follows from (72) that

$$K_{12} = K \tag{75}$$
$$k_0 = Kt^2(0)/(-s_a^2) \tag{76}$$
$$k_{12} = Kt(s_a)t(-s_a)/(-2s_a^2). \tag{77}$$

The relation analogous to (5) reads

$$(z_{11} - Z_1)(z_{22} + Z_2) = z_{12}^2 = \frac{K^2 t_+^2 t_-^2}{s^2(s^2 - s_a^2)^2}, \tag{78}$$

with

$$(z_{11} - Z_1) = \frac{K^2 t_+^2 t_-^2 h}{s(s^2 - s_a^2)Q} = \frac{K^2 t_+ t_-^2 h}{s(s^2 - s_a^2)q} \tag{79}$$

and

$$(z_{22} + Z_2) = \frac{Q}{s(s^2 - s_a^2)h} = \frac{t_+ q}{s(s^2 - s_a^2)h}. \tag{80}$$

From the last two forms we see that the augmentation process (ordinarily required in the Darlington procedure) actually need not be carried out, since the polynomial $t(s)$ cancels out again in the expression for $(z_{11} - Z_1)$.

For the determination of the polynomial $h(s)$ we will use the following slightly abbreviated persentation (which applies to the RC case also, of course). From (79) we form the rational function h/q and expand it into partial fractions, as follows

$$\frac{h(s)}{q(s)} = \frac{s(s^2 - s_a^2)(z_{11} - Z_1)}{K^2 t_+ t_-^2} = \frac{1}{K^2} \sum_{\nu=1}^{n} m_\nu \left(\frac{s_\nu^2 - s_a^2}{s - s_\nu}\right) \tag{81}$$

where

$$m_\nu = \frac{-s_\nu k_\nu}{t(s_\nu)t(-s_\nu)}, \tag{82}$$

and k_ν are the residues of Z_1 in its poles (which are the zeros of $q(s)$ at $s = s_1, s_2, \cdots s_n$). From the fact that z_{11} has a simple pole at $s = \infty$, the intermediate expression in (81) shows that h/q behaves like $1/s^2$ for $s \to \infty$. Hence $h(s)$ has the degree $n-2$, and the sum of the residues of h/q must be zero, or

$$\sum_{\nu=1}^{n} m_\nu (s_\nu^2 - s_a^2) = 0 \tag{83}$$

as expressed by (17) in the RC case. Thus the value of s_a^2 is determined from

$$s_a^2 = \frac{\sum m_\nu s_\nu^2}{\sum m_\nu}, \tag{84}$$

and we find for the constant multiplier of the polynomial $h(s)$

$$H = \frac{1}{K^2}\sum_{\nu=1}^{n} m_\nu s_\nu (s_\nu^2 - s_a^2). \tag{85}$$

Since terms in these sums are conjugate complex pairs we see that s_a^2 must have a real value and hence that (81) shows z_{11} to have real values for real values of the complex frequency variable s (since $Z_1(s)$, $t(s)$, $t(-s)$ have this property). The residues K_{11}, k_0, and k_{11} of z_{11} in (73) must, therefore, be real; and hence z_{11} has an identically zero real part for $s = j\omega$. From (80) and (81) we then see that

$$\frac{1}{z_{22}+Z_2} = \frac{s(s^2 - s_a^2)h}{t_+ q} = \frac{s^2(s^2 - s_a^2)^2(z_{11} - Z_1)}{K^2 t_+^2 t_-^2} \tag{86}$$

is analytic in the right half-plane inclusive of the j-axis where it has a positive real part. Hence this function is positive real, and so $z_{22}+Z_2$ must be positive real. It follows that its poles at $s = \pm s_a$ must lie on the j-axis (s_a^2 must be negative real). The poles of $z_{22}+Z_2$ at $s=0$, $s = \pm s_a$ and $s = \infty$ belong to z_{22}; the rest characterize Z_2, and we see that both z_{22} and Z_2 are pr functions. It follows that the residues k_0, K_{22}, and k_{22} of z_{22} are positive real; (76) yields a positive value for K, whereupon $K_{11} = K_{12}^2/K_{22}$ and $k_{11} = k_{12}^2/k_{22}$ are likewise shown to be positive real. The realizability of the zero-section as well as of the remainder function are assured.

The Brune cycle is simpler since the transmission zeros are a pair of conjugate points on j-axis defined by

$$t(s) = s^2 + \omega_0^2. \tag{87}$$

The lossless two terminal-pair network is characterized by the impedances

$$z_{12} = \frac{Kt(s)}{s} = \frac{k_0}{s} + k_{12}s \tag{88}$$

$$z_{11} = \cdots \frac{k_0}{s} + k_{11}s \tag{89}$$

$$z_{22} = \cdots = \frac{k_0}{s} + k_{22}s \tag{90}$$

in which we have

$$k_0 = Kt(0) = K\omega_0^2; \qquad k_{12} = K. \tag{91}$$

The given input impedance (minus its minimum real part at $\omega = \omega_0$) is again assumed in the form expressed by (70) (however, no augmentation is called for), and instead of (78) we have

$$(z_{11} - Z_1)(z_{22} + Z_2) = z_{12}^2 = \frac{K^2 t^2(s)}{s^2}. \tag{92}$$

Here we can immediately make the partial fraction expansion

$$\frac{1}{z_{22}+Z_2} = \frac{s^2(z_{11} - Z_1)}{K^2 t^2(s)} = \frac{1}{K^2}\sum_{\nu=1}^{n} \frac{-s_\nu^2 k_\nu}{t^2(s_\nu)} \times \frac{1}{s - s_\nu} \tag{93}$$

since the factor $(z_{11} - Z_1)$ must contain $t^2(s)$, and hence the function $(z_{22}+Z_2)^{-1}$ contains only the poles of $Z_1(s)$, the residues of which, as before, are denoted by k_ν. Since $Z_2(s)$, like $Z_1(s)$, is to have no j-axis poles, we get by considering (93) for $s \to 0$ and $s \to \infty$ respectively

$$\frac{1}{k_0} = \frac{1}{K^2}\sum_{\nu=1}^{n} \frac{k_\nu}{t^2(s_\nu)} = \frac{1}{K\omega_0^2} \tag{94}$$

and

$$\frac{1}{k_{22}} = \frac{1}{K^2}\sum_{\nu=1}^{n} \frac{-s_\nu^2 k_\nu}{t^2(s_\nu)}. \tag{95}$$

Terms in these sums are conjugates, so that the quantities k_0, K and k_{22} are real. It follows that k_{12}, (91), and $k_{11} = k_{12}^2/k_{22}$ are likewise real and hence that the real part of $z_{11}(j\omega)$ is identically zero. The remaining argument showing that the lossless zero-section is realizable and that $Z_2(s)$ is positive real, parallels that given for the Darlington cycle.